Dr. Marques Sophie
Office 519
Linear algebra
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marques@cims.nyu.edu

Problem Set # 9

Exercise 1($\star$):
If A is of the form

$$A = \left(\begin{array}{c|c} B & D \\ 0 & C \end{array} \right)$$

where B and C are two matrices, prove that

$$det(A) = det(B)det(C)$$

Exercise 2($\star$):
Prove that there is a basis for K^2 so that the matrix of the operator $L_A : K^2 \to K^2$, defined by

$$A = \begin{pmatrix} 2 & 4 \\ -1 & -2 \end{pmatrix}$$

over $\mathbb{C}$ and $\mathbb{R}$ becomes

$$\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$$

Exercise 3($\star$):

1. The matrix

$$A = \begin{pmatrix} cos(\theta) & -sin(\theta) \\ sin(\theta) & cos(\theta) \end{pmatrix}$$

   (θ real) yields an operator $L_A : \mathbb{R}^2 \to \mathbb{R}^2$ that you will recognize as a rotation (counter clock wise) about the origin by θ radians. Describe its eigenspaces over $\mathbb{R}$ and over $\mathbb{C}$.

2. Is this matrix diagonalizable over $\mathbb{R}$, over $\mathbb{C}$?

$$A = \begin{pmatrix} 3\sqrt{3} & -3 \\ 3 & 3\sqrt{3} \end{pmatrix}$$

   Determine all complex eigenvalues of A. If so, find a basis in $\mathbb{C}^2$ that diagonalizes $L_A : \mathbb{C}^2 \to \mathbb{C}^2$.

Note: You can give the complex eigenvalue in polar form. That is in the form:

$$\lambda = re^{i\theta} = (rcos(\theta)) + i(rsin(\theta))$$

with $r \geq 0$, $\theta \in \mathbb{R}$.

Exercise 4($\star$):
If $dim_K(V) = n$ and $T : V \to V$ has n distinct eigenvalues in K then T is diagonalizable, so V is the direct sum $\oplus_{i=1}^{n} E_{\lambda_i}$ of 1-dimensional eigenspaces.

Exercise 5($\star$):
Suppose $A \in M_n(\mathbb{C})$. If A is symmetric $A = A^t$ with complex entries, does L_A always have a diagonalizing basis? Prove or give a counterexample.
Hint: Try some symmetric 2×2 matrices $A = \begin{pmatrix} a & b \\ b & c \end{pmatrix}$.
Note: (context) If A is self-adjoint in $M_n(\mathbb{C})$ so that

$$A^* = A$$

(where $(A^*)_{ij} = \bar{A}_{ji}$ for all i, j) it is well known that there is a basis in K^n that diagonalizes the corresponding linear operator $L_A : K^n \to K^n$. (In fact there is even a basis that is ortonormal with respect to the standard inner product in K^n.).
Now, if we suppose that A is only symmetric with

$$A^t = A$$

(where $(A^t)_{ij} = A_{ji}$, for all i, j), when $A \in M_n(\mathbb{R})$ symmetry is the same thing as self adjointness and symmetry $A^t = A$ suffices to guarantee orthonormal diagonalizability of L_A. In this exercise we see what happens when $A \in M_n(\mathbb{C})$ only symmetric.